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On Certain Expansions of Elliptic, Hyperelliptic and Related Periodic Functions.[*]

By F. R. Moulton.

§ 1. Introduction.

The functions treated in this paper are defined by differential equations of the types which arise in many physical problems. For this reason special interest attaches to the real solutions. The problem of finding the properties of the solutions and their expansion is, however, one of analytic functions, and therefore it is not convenient to limit the discussion to real values of the variables. Consequently the analytic character of the solutions is determined and, because of the possible practical applications, especial attention is given to developing convenient methods for constructing them.

When the differential equations treated here are suitably specialized, they define certain elliptic or hyperelliptic functions. One of the most perfect theories in analysis is unquestionably that of elliptic functions. Nothing could be more systematic and elegant than the development of the properties of these functions when they are defined by the conditions that they shall be uniform and without essential singularities for all finite values of the independent variable, and that they shall have an algebraic addition theorem or that they shall be doubly periodic. These strong conditions make possible the detailed development of the theory, but they prevent its ready extension to more general cases. In this paper the point of view is altogether different. The functions are defined by differential equations, and the loss suffered in not explicitly prescribing their properties is partly balanced by the greater elasticity of the methods.

In physical problems the questions of periodicity of motion and the asymptotic approach of the dependent variables to limiting values are of primary interest. In the case of elliptic functions there can not be more than two periods, only one of which can be real; but in the case of hyperelliptic and more general abelian functions the number of real periods is not limited. In § 2 the question of the existence of the real and purely imaginary periods is treated. In this connection the qualitative character of the real curves defined by the differential equation is considered. In § 3 a number of special cases are treated, the simplest of which gives rise to the ordinary elliptic functions. In § 4 a method is given for expressing the periods by converging series in certain parameters of the differential equations. In § 5 the explicit construction of the periodic solutions from the differential equation is treated, and some of the important properties of the solutions are derived. In § 6 a method is given for constructing the solution from the first integral of the differential equations. In § 7 a method is given for constructing the solution, its properties being known from the discussion of § 5, from the values of the dependent variable and its derivatives for special values of the independent variable. The whole problem of constructing the solutions is there reduced simply to solving sets of ordinary linear algebraic equations. In § 8 application is made to the Legendre elliptic functions, and new expansions are given for them which, in § 9, are proved to converge for all finite real values of the independent variable if the absolute value of the modulus $\varkappa^2$ is less than unity. The series are very convenient for practical use and are developed to the twelfth power of $\varkappa$ inclusive.

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[*] Read before the American Mathematical Society under a different title, April 28, 1911.

§ 2. THE DIFFERENTIAL EQUATIONS AND PERIODS OF THE SOLUTIONS.

Consider the differential equation

$$\frac{d^2x}{dt^2} = x'' = \sum_{=-\infty}^{+\infty} A_j x^j, \tag{1}$$

where the A_j are constants. Suppose the series converges if $r < |x| < R$. This problem includes that of the motion of a particle moving subject to a central force varying as any positive or negative power of the distance. The determination of the motion of a particle in the equatorial plane of an oblate or prolate spheroid, or in fact in the equatorial plane of any body whose surface can be expressed by zonal harmonics, depends upon an equation of the type of (1). Equation (1) also includes all those problems whose solutions are obtained from the inversion of elliptic and hyperelliptic integrals. Reference to any treatise on elliptic functions will show the abundance of these problems.

Equation (1) admits the integral

$$x' = \sqrt{f(x)} = \sqrt{\sum_{j=-\infty}^{+\infty} B_j x^j + B \log x}, \tag{2}$$

where B_0 is the constant of integration and where

$$B = 2A_{-1}, \qquad B_j = \frac{2A_{j-1}}{j}, \quad j = -\infty, \ldots, +\infty, \text{ except } 0.$$

The series $f(x)$ also converges if $r < |x| < R$.

It follows from Cauchy's fundamental theorem on analytic differential equations that if, for any finite t_1, the dependent variable x takes such a value x_1 that the right member of (1) is regular, the solution of (1) is a regular function of t in the vicinity of t_1. Therefore the solution of (1) is regular for at least all finite values of t for which x stays in the ring $r < |x| < R$. It does not, of course, follow that for given initial conditions finite values of t exist such that x takes all values in the ring.

Suppose the roots of $f(x) = 0$ in the ring $r < r_1 \leqq |x| \leqq R_1 < R$ are $\alpha_1, \ldots, \alpha_p$, where p is finite except in the trivial case $f(x) \equiv 0$. Suppose the α_j are all distinct. Now join the circumferences r and R by a cut C which does not pass through an α_j and which has no double point. From (2) we have

$$t - t_1 = \int_{x_1}^{x} \frac{dx}{\sqrt{f(x)}}. \tag{3}$$

It is well known from the general theory* that the values of t obtained by integrating from $x = x_1$ back to $x = x_1$, over any path not passing through an α_j or crossing r_1, R_1 or C, are linear functions with integral coefficients of $k-1$ primitive periods $\omega_2, \ldots, \omega_p$ and I_1, where $\omega_2, \ldots, \omega_p$ are the values of the integrals along closed paths around α_1 and α_2, α_1 and α_3, $\ldots$, α_1 and α_p respectively, and where I_1 is the integral from 0 to α_1. These periods may not all be distinct and they are in general complex.

We shall confine our attention now to the real and purely imaginary periods. Suppose the coefficients of (1) and (2) are all real. Suppose $\alpha_1, \ldots, \alpha_m$ are the real positive roots of $f(x) = 0$, and that $\beta_1, \ldots, \beta_n$ are the real negative roots of $f(x) = 0$ satisfying the inequalities $r_1 \leqq |x| \leqq R_1$. It follows from the hypotheses above that $f(x)$ is finite, continuous, and single-valued in the real intervals $-R_1 < x < -r_1$, $r_1 < x < R_1$, and it changes sign only at the α_j and β_j.

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* Puiseux, *Journal de Mathématiques*, 1850, and Picard, *Traité d'Analyse*, Vol. II, Chap. VIII.

Suppose $f(x_1) > 0$, where $\alpha_i < x_1 < \alpha_{i+1}$, and that $f(x)$ does not vanish for $\alpha_i < x < \alpha_{i+1}$. Therefore x_1' is real and we may suppose it is positive. The solution x will be a real function of t and x will increase with increasing real values of t until it reaches α_{i+1}. It follows from (3) that this value will be attained for a finite value of t if α_{i+1} is a simple root of $f(x) = 0$. If $x = \alpha_{i+1}$ is a multiple root, x will approach α_{i+1} asymptotically as t becomes infinite. Suppose α_{i+1} is a simple root and that x has attained this value at t_{i+1}. It can not pass beyond this value, for then $f(x)$ would become negative and x' imaginary, whereas x is a regular function of t having real coefficients in the vicinity of this point. For the same reason it can not become complex as t takes increasing real values. It can not become a constant at this point, for x'' is distinct from zero, since α_{i+1} is a simple root of $f(x) = 0$. Therefore as t increases beyond t_{i+1}, x' becomes negative and x decreases continuously to α_i, which it reaches in a finite time if α_i is a simple root of $f(x) = 0$. Then the sign of x' changes and x increases to α_{i+1} again. That is, under the conditions set down x oscillates between α_i and α_{i+1} in the period

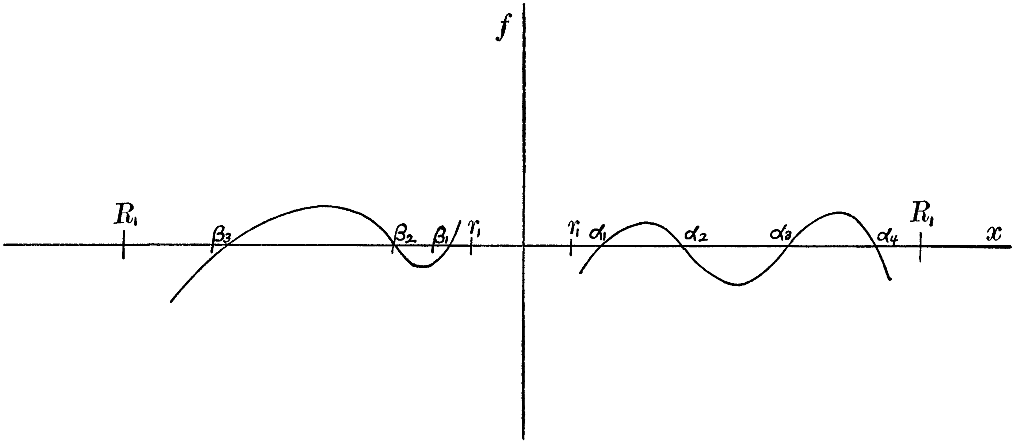

FIG. 1.

$$T(\alpha_i, \alpha_{i+1}) = 2\int_{\alpha_i}^{\alpha_{i+1}} \frac{dx}{\sqrt{f(x)}}. \tag{4}$$

Similar results are true for all other closed intervals in the ring whose end-points are simple real roots of $f(x) = 0$ for which $f(x)$ is positive. The periods are not necessarily distinct.

In an interval in which $f(x)$ is negative we make the transformation

$$t = \sqrt{-1}\,\tau,$$

after which the problem in τ has the same properties it previously had in t.

That is, there are purely imaginary periods corresponding to the closed intervals in which $f(x)$ is negative, and for them x is real.*

The solution involves two arbitrary constants, one of which is B_0; the other is additive to t, since (1) is unchanged by the transformation $t = t_0 + \tau$. The constant t_0 has no influence upon the analytic character of the solutions, but it is quite otherwise with B_0. Since this constant is additive in $f(x)$, changing it simply produces a vertical shift in the curves of Fig. 1. Suppose B_0 is increased. As a minimum approaches the x-axis, the two real periods associated with the adjacent intervals for which $f(x)$ is positive approach infinity. On the other hand, the associated imaginary period approaches a finite determinate limit, as can be shown by forming the expression for the period and passing to the limit. If α_{i+1} and α_{i+2} are the two roots which approach equality, we have

$$f(x) = -(x-\alpha_{i+1})(\alpha_{i+2}-x)\,\psi(x),$$

where $\psi(x) > 0$ for $\alpha_{i+1} \overset{=}{<} x \leqq \alpha_{i+2}$. Let the value at which α_{i+1} and α_{i+2} unite be α and $\psi(\alpha) = A > 0$. Let $x = \alpha + \xi$. When $\alpha_{i+2} - \alpha_{i+1}$ is sufficiently small, $\psi(\alpha + \xi)$ can be expanded in a power-series in ξ which converges for all x in $\alpha_{i+1} \overset{=}{<} x \leqq \alpha_{i+2}$. Therefore the imaginary period is

$$\sqrt{-1}\,T = \frac{2}{\sqrt{A}}\int_{\alpha_{i+1}-\alpha}^{\alpha_{i+2}-\alpha} \frac{[1+\phi'(\alpha)\xi+\ldots.]\,d\xi}{\sqrt{(\alpha-\alpha_{i+1})(\alpha_{i+2}-\alpha)+(\alpha_{i+1}+\alpha_{i+2}-2\alpha)\xi-\xi^2}}, \tag{5}$$

where $\phi = \dfrac{\sqrt{A}}{\sqrt{\psi}}$. The limit of this integral as $\alpha_{i+1} = \alpha$, $\alpha_{i+2} = \alpha$ is $\dfrac{2\pi}{\sqrt{A}}$. As B_0 increases beyond the value for which $\alpha_{i+1} = \alpha_{i+2}$, the purely imaginary period and one of the real periods giving real values of x disappear, and there remains in place of them a single period, which is real, giving real values of x. If the value of B_0 decreases until a maximum of $f(x)$ crosses the x-axis, the rôles of the real and imaginary periods are interchanged.

Suppose the positive roots of $f(x) = 0$ in the interval $r_1 < |x| < R_1$ are $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Suppose $f(x)$ is positive in the intervals $\alpha_1\alpha_2$, $\alpha_3\alpha_4$ and negative in $\alpha_2\alpha_3$. In the positive intervals x is real for real t, and in the negative intervals it is real for purely imaginary values of $t = \sqrt{-1}\,\tau$. In Fig. 2 these curves are given. Those which belong to real values of t are given in full lines and that which belongs to imaginary values of t is given in chain dots. In its case the t-axis should be read the τ-axis.

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* While elliptic functions can have only two periods whose ratio can not be real, the functions defined by these more general equations can have any number of periods. This point has been insisted on by Casorati in two memoirs in *Acta Mathematica*, Vol. VIII, 1886.

Since the solutions carry an arbitrary constant added to t, they may be displaced in the horizontal direction without changing their character. The figure has been drawn so that x starts at α_1 at $t = 0$. It arrives at α_2 at $t = \frac{1}{2} T_{12}$ and returns to α_1 at $t = T_{12}$. But if t takes purely imaginary increments when it arrives at $\frac{1}{2} T_{12}$, x will describe the curve belonging to the interval $\alpha_2 \alpha_3$. At $t = \frac{1}{2} T_{12} + \sqrt{-1}\, \frac{1}{2} T_{23}$, x will have the value α_3, and will return to α_2 at $t = \frac{1}{2} T_{12} + \sqrt{-1}\, T_{23}$. But if t takes real increments after $t = \frac{1}{2} T_{12} + \sqrt{-1}\, \frac{1}{2} T_{23}$, x will describe the curve belonging to the interval $\alpha_3 \alpha_4$, and will arrive at α_4 at $t = \frac{1}{2} T_{12} + \sqrt{-1}\, \frac{1}{2} T_{23} + \frac{1}{2} T_{34}$.

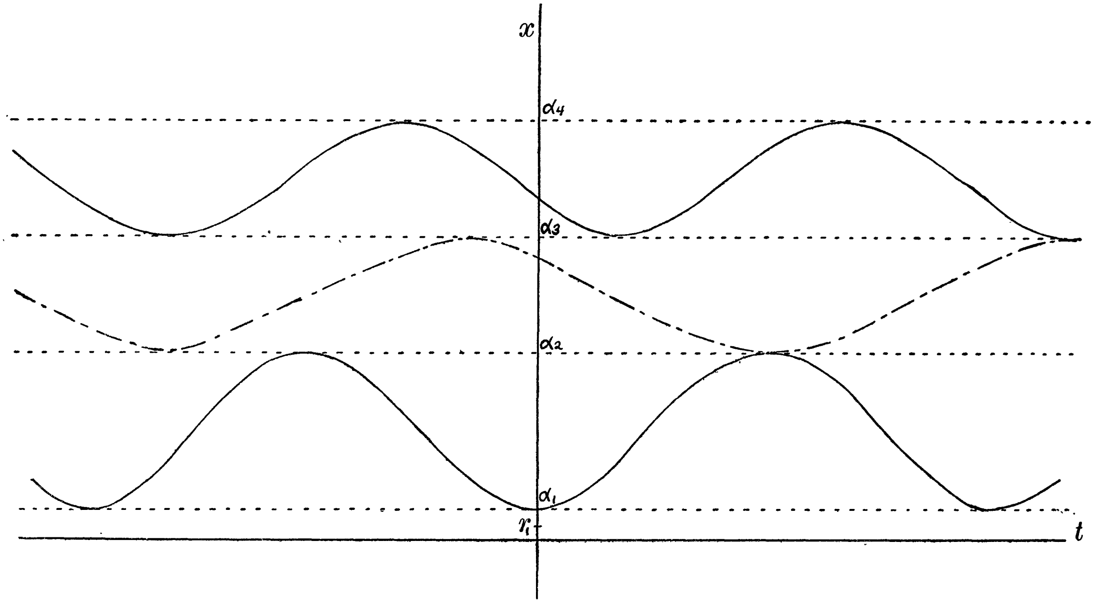

FIG. 2.

The discussion of the branches for $|x| > R$ can not be made unless more properties of $f(x)$ are given.

§ 3. SOME SPECIAL CASES.

CASE 1. $B = 0$ *and only a finite number of the* B_{-j} *distinct from zero.*

If only a finite number of the B_{-j} are distinct from zero, $r = 0$. If B is distinct from zero, the constant B_0 is not defined for $x = 0$. If x has initially some value distinct from zero, and if for $t = t_1$ it becomes zero, t_1 is an essential singular point. If $B_{-j} = 0$ for $j = 1$ and $j > m$, and if $B_{-m} \neq 0$, the nature of the function in the vicinity of $t = t_1$, where $x = 0$ for $t = t_1$, can easily be determined from (2). We have at once

$$\left.\begin{aligned} t - t_1 &= \int_0^x \frac{x^{\frac{m}{2}}\,dx}{\sqrt{B_{-m} + B_{-m+1}x + \cdots}} \\ &= \frac{2}{2+m}\,\frac{x^{\frac{m+2}{2}}}{\sqrt{B_{-m}}}\,[1 + C_1 x + \cdots]. \end{aligned}\right\} \tag{6}$$

Inverting this series, we have

$$x = \left[\frac{(2+m)^2}{4} B_{-m}(t-t_1)^2\right]^{\frac{1}{m+2}} \left\{1 + D_1 (t-t_1)^{\frac{1}{m+2}} + \cdots\right\}. \tag{7}$$

If the constants $A_{\pm j}$ are real, the coefficients D_i in (7) are real. The variable x considered as a function of $t - t_1$ has $m + 2$ determinations which vanish at $t - t_1 = 0$. If m is even, they are all complex, or two are real and m of them are complex for both positive and negative real values of $t - t_1$ according as B_{-m} is negative or positive. If B_{-m} is negative, two branches are real for both positive and negative purely imaginary values of $t - t_1$. If m is odd, there are $m + 1$ complex values of x and one real value for both positive and negative real values of t. In this case there is one real value of x for both positive and negative purely imaginary values of t.

Suppose m is even, B_{-m} is positive, and $f(x)$ has simple zeros at $x = \alpha_1, \alpha_2, \beta_1, \beta_2$, where $\alpha_1 < \alpha_2 < R$, $-\beta_1 < -\beta_2 < R$. Suppose $x = 0$ at $t = t_1$. There are two real branches in the neighborhood of $t = t_1$ for both positive and negative real values of $t - t_1$. Let us take $t - t_1$ positive and first consider the branch for which x' becomes positive. It follows from (2) that on this branch x' stays positive and x increases until it reaches the value α_1. For real increasing values of $t - t_1$, x' changes sign and x decreases to zero, which it reaches at a finite value of $t - t_1 = T(0, \alpha_1)$. At this value of $t - t_1$, x' changes sign by passing through infinity and x remains positive after passing through zero. Then the cycle to α_1 and return is repeated. There is a similar cycle to β_1 and return with the period $T(0, \beta_1)$ obtained by starting with the branch for which x' became negative for real $t - t_1$ near zero.

If $t - t_1$ takes purely imaginary increments after arriving at any odd multiple of $\frac{1}{2}\,T(0, \alpha_1)$ on the branch which lies between 0 and α_1, x will describe a real curve between α_1 and α_2; and starting from the other branch there is similarly a real curve between β_1 and β_2 for purely imaginary increments to $t-t_1$ after it arrives at any multiple of $T(0, \beta_1)$. Fig. 3 shows the curves for real $t-t_1$ in full lines, and for imaginary $t-t_1$ in chain dots. The curves are drawn satisfying the initial condition $x=0$ at $t=0$, and by continuous variation of t from this value all the branches shown can be obtained.

Now suppose that m is again even, that $f(x)$ has simple zeros at $x=\alpha_1, \alpha_2, \beta_1, \beta_2$, but that B_{-m} is negative. The only qualitative difference between this case and that just treated is that the full curves belonging to real $t-t_1$, and the chain-dot curves belonging to a purely imaginary $t-t_1$, are interchanged in position.

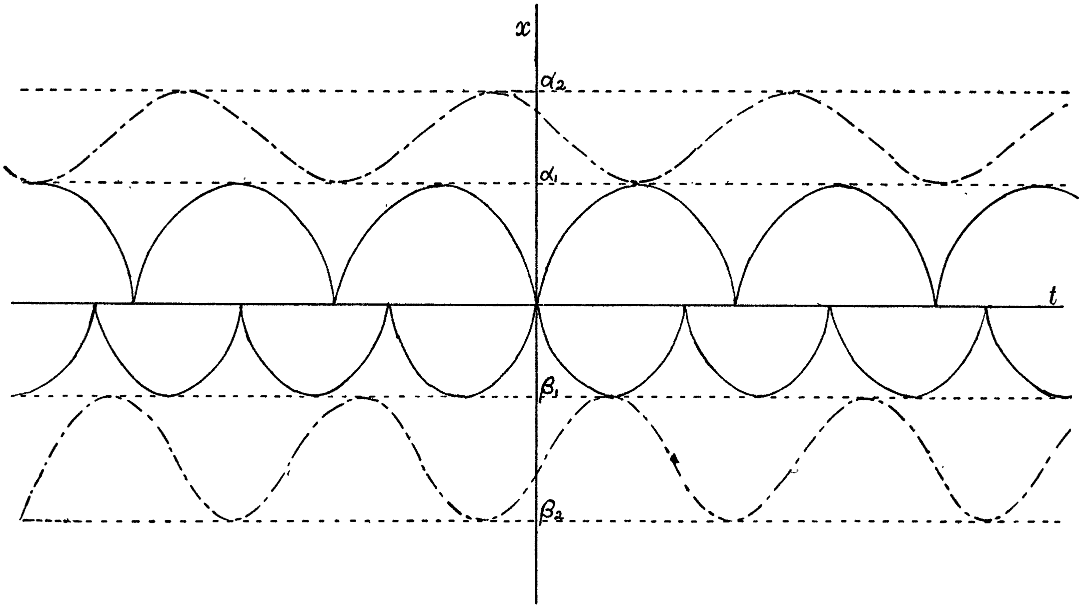

FIG. 3.

Suppose m is odd, B_{-m} is again positive, and $f(x)$ has simple zeros at $x=\alpha_1, \alpha_2, \beta_1, \beta_2$, where $\alpha_1 < \alpha_2 < R$, $-\beta_1 < -\beta_2 < +R$. In this case there is only one real value of x both when $t-t_1$ is positive and when it is negative, and in both cases x is positive. If $t-t_1$ is a positive or negative pure imaginary, there is one real branch on which x is negative. For real $t-t_1$, whether increasing or decreasing from $t-t_1=0$, x increases to α_1 and then returns to zero after a finite time $T(0, \alpha_1)$. For purely imaginary $t-t_1$, x decreases to β_1 and returns to zero after the finite time $\sqrt{-1}\, T(0, \beta_1)$. If $t-t_1$ increases by purely imaginary increments from $\frac{1}{2}T(0, \alpha_1)$, x remains real and increases to α_2, and then, decreasing, returns to α_1 at the finite time $\sqrt{-1}\,T(\alpha_1, \alpha_2)$. If $t - t_1$ increases by real increments from $\sqrt{-1}\,\frac{1}{2}\,T(0, \beta_1)$, x remains real and decreases to β_2 and then returns to β_1 after the finite interval $T(\beta_1, \beta_2)$. Fig. 4 gives the real curves with the arbitrary determined so that $x = 0$ at $t = 0$. By continuous variation of t through real and purely imaginary values over the proper ranges, all the curves will be described.

The case where B_{-m} is negative differs qualitatively only in that the curves for real and purely imaginary values of $t - t_1$ are interchanged in position.

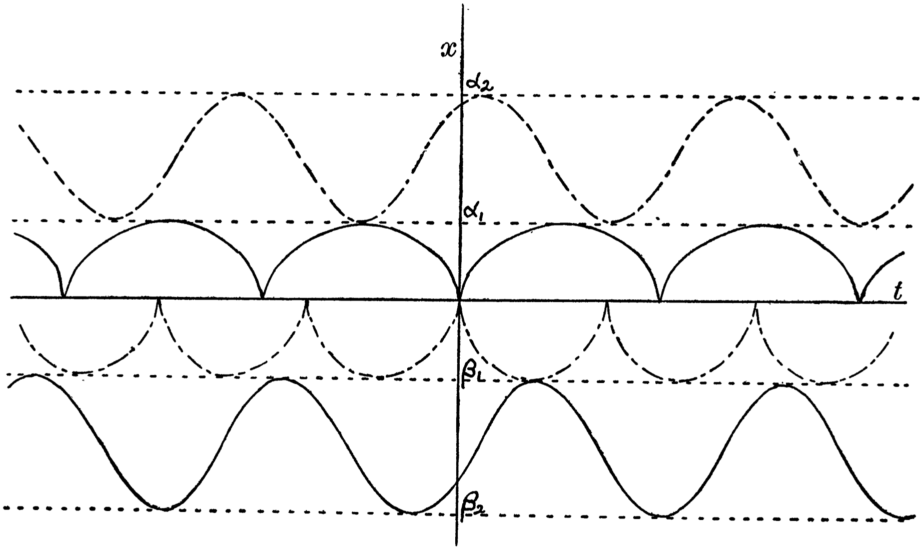

FIG. 4.

CASE 2. $B = 0$ *and only a finite number of the* B_j *distinct from zero.*

We now consider the case where $B = 0$, $B_n \neq 0$ and the $B_j = 0$, $j > n$. In this case $R = \infty$. The question of chief interest is the nature of x as a function of t for those values of t for which $x = \infty$. Let $x = 1/\xi$ and (2) becomes

$$\xi' = -\frac{\sqrt{\cdots + B_{-m}\xi^{m+n} + \cdots + B_{-1}\xi^{n+1} + B_0\xi^{n} + \cdots + B_n}}{\xi^{\frac{n-4}{2}}}. \tag{8}$$

If $n \leqq 4$, the case of elliptic or exponential functions, ξ is a regular function of t at t_1, where $\xi = 0$, $x = \infty$ at $t = t_1$. In other cases we find from

$$-(t - t_1) = \int_0^{\xi} \frac{\xi^{\frac{n-4}{2}}\, d\xi}{\sqrt{B_n + B_{n-1}\xi + \cdots}} \tag{9}$$

that

$$\xi = \left[\frac{(n-2)^2}{4} B_n (t - t_1)^2\right]^{\frac{1}{n-2}} \left\{1 + \cdots\right\}. \tag{10}$$

The discussion of the real branches of the curves defined by this equation is the same as that of (7).

Suppose α is the largest positive root of $f(x)=0$ and that β is the negative root having the largest modulus. Suppose further that these roots are simple. Then it follows from (9) that ξ will vary from 0 to $1/\alpha$ or to $1/\beta$ in a finite interval of time $T(\alpha, \infty)$ or $T(\beta, \infty)$. Under the hypotheses which have been adopted, if $t-t_1$ increases continuously beyond $T(\alpha, \infty)$ or $T(\beta, \infty)$, by real increments if B_n is positive and by purely imaginary increments if B_n is negative, ξ' will change sign and ξ will return to zero. For B_n positive and n even the infinite branches of the real curves are given in Fig. 5.

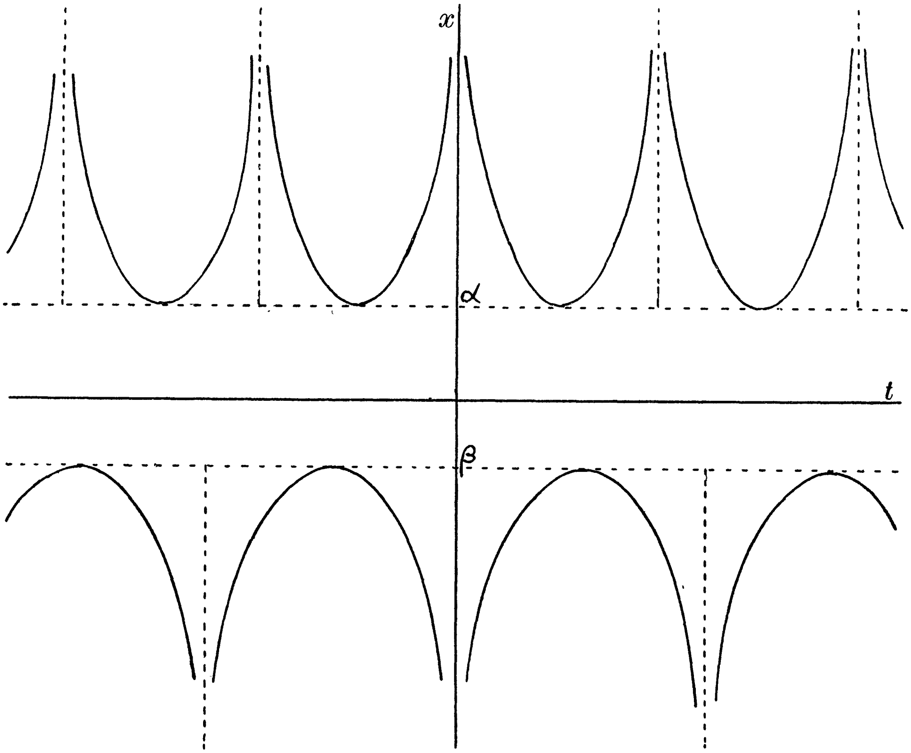

FIG. 5.

If B_n were negative, the curves would be similar for purely imaginary values of t.

If n is odd and B_n positive, there are positive real branches for real values of $t-t_1$, and negative real branches for purely imaginary values of $t-t_1$. Qualitatively the figure for the real branches would be the same as Fig. 5, except that the negative branches would belong to purely imaginary values of $t-t_1$. If B_n were negative, the results would differ qualitatively from those where B_n is positive only in that the real and purely imaginary values of $t-t_1$ would be interchanged in position.

CASE 3. *$f(x)$ a polynomial of the fourth or third degree.*

Suppose the roots of $f(x)=0$ are $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and that the coefficient of x^4 is unity. Suppose $\alpha_1, \ldots, \alpha_4$ are real and distinct and that the notation has been so chosen that $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$. We now make the linear fractional transformation

$$\frac{x-\alpha_3}{x-\alpha_2} \cdot \frac{\alpha_4-\alpha_2}{\alpha_4-\alpha_3} = \frac{\xi-\xi_3}{\xi-\xi_2} \cdot \frac{\xi_4-\xi_2}{\xi_4-\xi_3}, \tag{11}$$

which transforms $\alpha_2, \alpha_3, \alpha_4$ into ξ_2, ξ_3, ξ_4 respectively, and any other point x forming a certain anharmonic ratio with $\alpha_2, \alpha_3, \alpha_4$ into a ξ forming the same anharmonic ratio with ξ_2, ξ_3, ξ_4. We assign the values $\xi_2=-1$, $\xi_3=+1$, $\xi_4=+1/\varkappa$, where $\varkappa$ is determined by the condition that the point α_1 shall be transformed into $-1/\varkappa$. This is, of course, the Legendre normal form. Then (11) becomes

$$\frac{(1+\varkappa)^2}{(1-\varkappa)^2} = \frac{\alpha_1-\alpha_3}{\alpha_1-\alpha_2} \cdot \frac{\alpha_4-\alpha_2}{\alpha_4-\alpha_3} = C > 1.$$

Solving, we get

$$\varkappa = \frac{\sqrt{C}-1}{\sqrt{C}+1} \text{ and } \varkappa = \frac{\sqrt{C}+1}{\sqrt{C}-1}. \tag{12}$$

The former of these two values of $\varkappa$ is less than unity and will be adopted. It leaves the transformed points in the same order on the real axis as the original ones. Then equation (2) becomes

$$\left.\begin{aligned} \xi' &= \sqrt{A}\sqrt{(1-\xi^2)(1-\varkappa^2\xi^2)}, \\ A &= \frac{(\alpha_2-\alpha_1)(\alpha_4-\alpha_3)}{(1-\varkappa)^2}. \end{aligned}\right\} \tag{13}$$

If $f(x)$ were of the third degree, that is, if α_4 were infinite, the transformation would lead to the same result.

In the case under consideration ξ is a regular function of t for all finite values of t. It has four periods,

$$T(-1/\varkappa, -1),\ T(-1, +1),\ T(+1, +1/\varkappa),\ T(1/\varkappa, +\infty, -\infty, -1/\varkappa).$$

The first and third are obviously equal. Since the transformation $\xi = -1/\varkappa\eta$ leaves the form of the first of (13) invariant, the second and fourth periods are also equal. One period is real and the other purely imaginary. The real curves are shown in Fig. 6.

CASE 4. *The differential equation* $x' = \sqrt[n]{(1-x^2)(1-\kappa^2 x^2)}$.

The case $n > 2$ will be discussed here only to show in what general respects the functions defined by this differential equation differ from those heretofore treated. For simplicity $f(x)$ is taken in the Legendre normal form. But it is noted that if we should start with $f(x)$ having the zeros $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, the transformation (11) would not in this case reduce the differential equation precisely to the normal form.

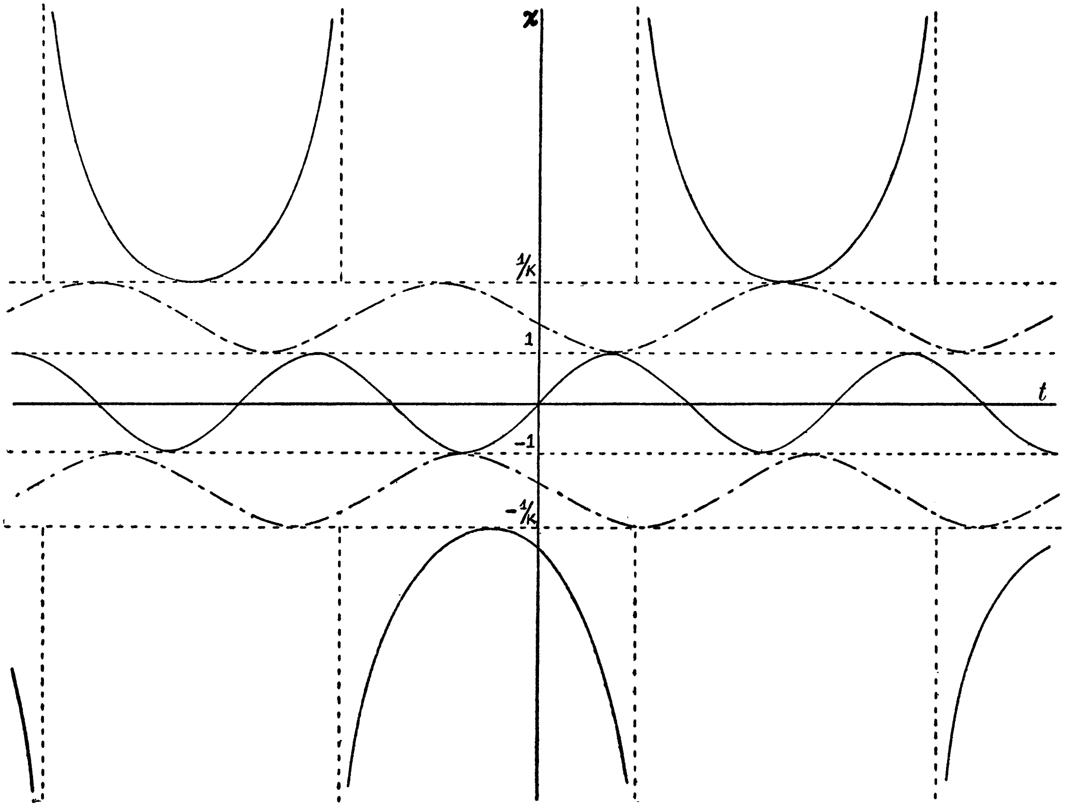

FIG. 6.

The solutions of

$$x' = \sqrt[n]{(1-x^2)(1-\kappa^2 x^2)} \tag{14}$$

are regular functions of t, certainly for all finite values, except possibly those for which $x = \pm 1$, $\kappa x = \pm 1$, $x = \infty$. We shall examine the function at these points in turn.

Suppose $x=0$ at $t=0$. One of the determinations of the radical is real and positive. If we choose this one and take t real and positive, x will increase to unity, which it will reach at the finite time

$$t_1 = T(0,1) = \int_0^1 \frac{dx}{\sqrt[n]{(1-x^2)(1-\kappa^2 x^2)}}. \tag{15}$$

If n were even and the negative determination of the radical had been taken, x would have arrived at -1 at the time t_1.

We must investigate the nature of x as a function of t in the vicinity of $t=t_1$. Let $x=1-\xi$; then (14) gives

$$t-t_1 = -\int_0^{\xi} \frac{d\xi}{\sqrt[n]{\xi(2-\xi)(1-\kappa^2+2\kappa^2\xi-\kappa^2\xi^2)}}. \tag{16}$$

Expanding the integrand as a series and integrating, we get

$$-(t-t_1) = \xi^{\frac{n-1}{n}}[a_0+a_1\xi+\ldots\ldots],$$

the a_i being real and a_0 positive. Inverting this series, we have

$$\left.\begin{aligned} \xi &= b_1\tau + b_2\tau^2 + \ldots\ldots, \quad b_1>0, \\ \tau &= \{[-(t-t_1)]^n\}^{\frac{1}{n-1}}. \end{aligned}\right\} \tag{17}$$

It follows from the second of these equations that τ is not changed by multiplying $t-t_1$ by any n-th root of unity. Therefore, there are n values of $t-t_1$ having the same modulus which give the same value of τ. As $t-t_1$ passes through zero, τ passes through zero; and its value after changing sign is, for a given modulus, equal to that before multiplied by $e^{\frac{n\pi\sqrt{-1}}{n-1}}$, which is a complex number except when $n=2$. Therefore, if $t-t_1$ is real any real curve belonging to it becomes complex when $t-t_1$ passes through zero. It is obvious therefore that these functions can not represent the motions of physical bodies.

If n is odd, ξ has two real values, one positive and one negative, for real negative values of $t-t_1$. In this case ξ is not real for any other real or purely imaginary values of $t-t_1$. If n is even, ξ has only one real value for real values of $t-t_1$, and this value is positive and the same whether $t-t_1$ is positive or negative with a given modulus. But in this case ξ is also real, though it is negative, for $t-t_1$ equal to a real positive or negative number multiplied by any n-th root of -1. If n is 2, these are the purely imaginary values of $t-t_1$ encountered in the problems heretofore considered. In general, if n is even and not congruent to zero mod 4, two of these values are conjugate pure imaginaries. But if n is a multiple of 4, all these values of $t-t_1$ are complex.

It follows from this discussion that if we start with $x=0$, x' real and positive at $t=0$, x will take the value $+1$ at a finite time $t=t_1$. At this point there are $n-1$ branches of the curves satisfying the differential equation, and there are values of $t-t_1$, real or complex, for which x remains real and increases, and others for which it remains real and decreases. On the branch on which it increases it arrives at $+1/\kappa$ at a finite time $t=t_2$. At this point the description of the branching differs from that at $x=+1$ only in that positive and negative as applied to ξ, where $x=1-\xi$ as before, must be interchanged. Consequently there is a real increasing value of x beyond $x=1/\kappa$ for real positive values of $t-t_2$. If we follow this curve, x will tend toward infinity, which it will reach at the time

$$T(1/\kappa,\infty)=\int_{1/\kappa}^{\infty}\frac{dx}{\sqrt[n]{(1-x^2)(1-\kappa^2x^2)}}, \tag{18}$$

which is infinite unless $n<4$.

Suppose $n=3$ and let $x=-1/\xi$. Then, if $x=\infty$ at $t=t_3$, we have

$$t-t_3=\int_0^{\xi}\frac{d\xi}{\xi^{2/3}\sqrt[3]{(1-\xi^2)(\kappa^2-\xi^2)}},$$

$$t-t_3=\xi^{1/3}[a_0+a_1\xi+\ldots.],$$

the a_i being real and a_0 positive. From this we get

$$\left.\begin{aligned}\xi&=b_0(t-t_3)^3+\ldots.,\\ x&=\frac{-1}{t)^3}\{c_0+c_1(t-t_3)+\ldots.\},\ c_0>0.\end{aligned}\right\} \tag{19}$$

The discussions for -1, $-1/\kappa$ and ∞ are essentially the same. We see that the details are quite different when n is even from those when it is odd, and that $x=\infty$ for finite values of t only if n is 2 or 3. In the former they are regular points, and in the latter poles of order 3. For $n=3$ the real curves are shown in Fig. 7.

§ 4. EXPLICIT COMPUTATION OF THE PERIODS.

Suppose it is desired to find the period corresponding to the interval α_i, α_{i+1}. One method is obviously to compute the numerical value of (4) by mechanical quadratures, as Legendre did in the case of elliptic integrals. But this is not sufficient in the present discussion, because in constructing the solutions in §§ 5–7 it will be necessary to have the periods developed in converging series.

Since the zeros of $f(x)$ in $r_1 \overset{=}{<} |x| \leqq R_1$ are $\alpha_1, \ldots, \alpha_p$, we may write

$$f(x) = (x - \alpha_1) \ldots (x - \alpha_p)\,\psi(x), \tag{20}$$

where $\psi(x)$ is finite and distinct from zero in the ring $r_1 \overset{=}{<} |x| \leqq R_1$. Suppose α_i and α_{i+1} are real and that $f(x_0) > 0$ for $\alpha_i < x_0 < \alpha_{i+1}$, so that the period is real.

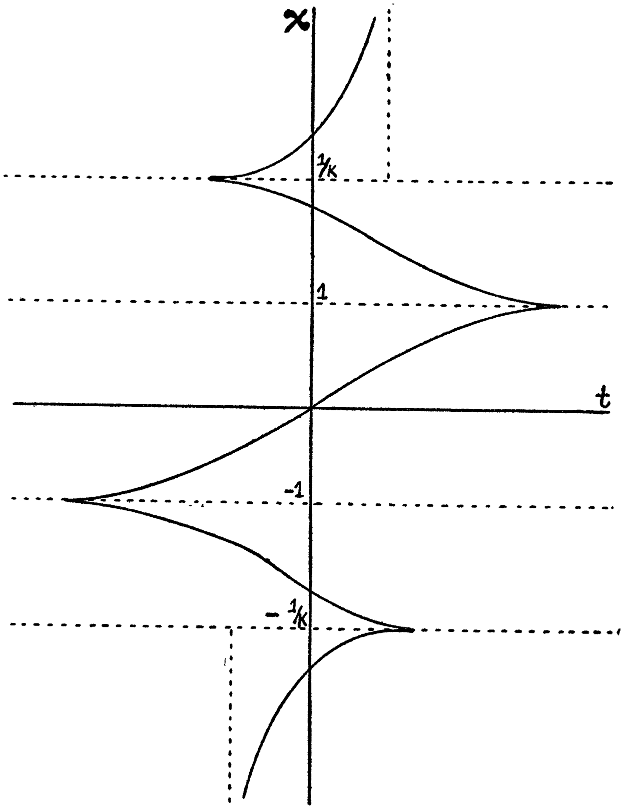

FIG. 7.

Let

$$x = \frac{\alpha_i + \alpha_{i+1}}{2} + \frac{\alpha_{i+1} - \alpha_i}{2}\,\xi. \tag{21}$$

Then (20) becomes

$$f(x) = (1 - \xi^2) \prod_{j=1}^{p} \left[\frac{\alpha_i + \alpha_{i+1} - 2\alpha_j}{2} + \frac{\alpha_{i+1} - \alpha_i}{2}\,\xi \right] \chi(\xi), \tag{22}$$

where $\chi(\xi)$ is finite and distinct from zero for all ξ corresponding to $r_1 \overset{=}{<} |x| \leqq R_1$. Those α_j which are complex occur in conjugate pairs. In these cases we take their products and obtain factors of the form $\beta_j\xi^2+\gamma_j\xi+\delta_j$, where β_j, γ_j and δ_j are real constants. Then we may write (22) in the form

$$f(x)=\frac{A^2(\alpha_{i+1}-\alpha_i)^2}{4}(1-\xi^2)(1+\rho_1\xi)$$
$$\dots\dots(1+\rho_k\xi)[1+\lambda_1\xi+\mu_1\xi^2]\dots\dots[1+\lambda_l\xi+\mu_l\xi^2]F(\xi), \quad (23)$$

where $\rho_1, \dots\dots, \rho_k, \lambda_1, \mu_1, \dots\dots, \lambda_l, \mu_l$ and the coefficients of $F(\xi)$ are real constants, and where $F(0)=1$. Moreover $\rho_1, \dots\dots, \rho_k$ are less than unity because α_i and α_{i+1} are consecutive real roots of $f(x)=0$.

In order to compute (4) we must take the square root of the reciprocal of $f(x)$. The square roots of the reciprocals of the factors $(1+\rho_1\xi)\dots\dots(1+\rho_k\xi)$ are expansible as power-series in ξ which converge for all $|\xi|\leqq 1$. The expansion of the square root of the reciprocal of $1+\lambda_j\xi+\mu_j\xi^2$ as a power-series in ξ converges if $|\xi|<1/\sqrt{|\mu_j|}$. We shall suppose at present that the $|\mu_j|$ are all less than unity. Then (4) becomes

$$T(\alpha_i, \alpha_{i+1})=\frac{2}{A}\int_{-1}^{+1}\frac{[1+P_1\xi+P_2\xi^2+\dots\dots]d\xi}{\sqrt{1-\xi^2}}, \quad (24)$$

where $P_1, P_2, \dots\dots$ are polynomials in $\rho_1, \dots\dots, \rho_k, \lambda_j, \mu_j$ and the coefficients of $F(\xi)$. The series under the integral sign converges for $|\xi|\leqq 1$. Since

$$\int_{-1}^{+1}\frac{\xi^{2n+1}}{\sqrt{1-\xi^2}}d\xi=0, \quad \int_{-1}^{+1}\frac{\xi^{2n}}{\sqrt{1-\xi^2}}d\xi=\frac{(2n)!\,\pi}{2^{2n}(n!)^2},$$

the period is

$$T(\alpha_i, \alpha_{i+1})=\frac{2\pi}{A}\left\{1+\sum_{n=1}^{\infty}\frac{(2n)!\,P_{2n}}{2^{2n}(n!)^2}\right\}. \quad (25)$$

Since

$$\frac{(2n)!}{2^{2n}(n!)^2}=\frac{1\,.\,3\,.\,5\,.\dots.\,2n-1}{2\,.\,4\,.\,6\,.\dots.\,2n}<1$$

and the integrand of (24) converges for $|\xi|<1$, the right member of (25) is a convergent series.

In the simple elliptic case

$$f(x)=(1-x^2)(1-\kappa^2x^2), \quad \kappa^2<1,$$

the real period is

$$T=2\pi\left\{1+\left(\frac{1\,.\,3}{2\,.\,4}\right)^2\kappa^2+\dots\dots+\left(\frac{1\,.\,3\,.\dots.\,2n-1}{2\,.\,4\,.\dots.\,2n}\right)^2\kappa^{2n}+\dots\dots\right\}. \quad (26)$$

Suppose now that $|\mu_j|>1$, but that all other $|\mu_i|<1$, and that

$$1+\lambda_j\xi+\mu_j\xi^2=\mu_j[\xi-\beta+\sqrt{-1}\,\gamma][\xi-\beta-\sqrt{-1}\,\gamma],$$

where β and γ are real. The integral in this case can be computed from -1 to β, and then from β to $+1$. Let

$$\xi = \frac{\beta - 1}{2} + \frac{\beta + 1}{2}\eta.$$

Then $\eta = -1$ when $\xi = -1$, and $\eta = +1$ when $\xi = \beta$. The quadratic factor $[(\xi - \beta)^2 + \gamma^2]$ becomes $\frac{(1+\beta)^2}{4}(1-\eta)^2 + \gamma^2$. The square root of its reciprocal is expansible as a converging power-series in η if $|\eta| < \sqrt{1 + \frac{4\gamma^2}{(1+\beta)^2}}$. Therefore the first of the two integrals is

$$T(\alpha_i, \beta) = \int_{-1}^{+1} \frac{[1 + Q_1\eta + Q_2\eta^2 + \ldots.]\,d\eta}{\sqrt{1-\eta}}.$$

For the second integral there is a similar transformation and expansion.

If two of the $|\mu_j|$ were greater than unity, the integral in x from α_i to α_{i+1} would be broken up in a similar way into the sum of three integrals which would be defined by converging series. This process of division can obviously be extended to include the case where any finite number of $|\mu_j|$ are equal to or greater than unity.

§ 5. EXPLICIT CONSTRUCTION OF THE PERIODIC SOLUTION.

Suppose it is desired to find the solution for which x oscillates between α_i and α_{i+1}. By the transformation (21) the differential equation becomes

$$\xi' = A\sqrt{(1-\xi^2)(1+\rho_1\xi)\ldots.(1+\rho_k\xi)[1+\lambda_1\xi+\mu_1\xi^2]\ldots.[1+\lambda_l\xi+\mu_l\xi^2]F(\xi)}, \quad (27)$$

where $F(\xi)$ is a power-series in ξ which converges if $|\xi| \leqq 1$.

For convenience let us make the transformation

$$\left.\begin{aligned} \rho_i &= \bar{\rho}_i\mu, \\ \lambda_i &= \bar{\lambda}_i\mu, \\ \mu_i &= \bar{\mu}_i\mu^2, \\ c_i &= \bar{c}_i\mu^i, \end{aligned}\right\} \quad (28)$$

where the c_i are coefficients of expansion of $F(\xi)$. The parameter μ will guide the computations; in numerical applications it will be taken equal to unity and the bars on the ρ_i, λ_i, μ_i and c_i will be omitted. After the transformations (28), equations (27) and (25) become

$$\left.\begin{aligned} \xi' &= A\sqrt{(1-\xi^2)\,P(\xi;\mu)}, \\ T &= \frac{2\pi}{A}\,Q(\mu), \end{aligned}\right\} \quad (29)$$

where P is a power-series in μ which converges if $|\xi| \leqq 1$, provided all $|\mu_j| < 1$, and where $Q(\mu)$ is the series (25). The coefficient of μ^i in P contains ξ^i as a factor.

In order to place in evidence the period in the construction of the solution let us make the transformation

$$t - t_0 = \frac{1}{A}\, Q(\mu)\, \tau. \tag{30}$$

Therefore, the period in the variable τ is 2π. If we denote by dots the derivatives with respect to τ, we have

$$\left.\begin{aligned} \dot{\xi} &= Q(\mu)\sqrt{(1-\xi^2)\,P(\xi;\mu)}, \\ \ddot{\xi} &= -\,Q^2(\mu)\,\xi\,P(\xi;\mu) + \tfrac{1}{2}\,Q^2(\mu)\,(1-\xi^2)\,\frac{\partial P}{\partial \xi}(\xi;\mu). \end{aligned}\right\} \tag{31}$$

By Poincaré's extension of Cauchy's theorem* the second of these equations can be integrated as a power-series in μ, and $|\mu|$ can be taken so small that the series will converge for any preassigned interval for τ. The same result is effected by taking ρ_i, λ_i, μ_i and c_i small, and μ equal to unity.

Let us construct the actual solution of the second of (31) with the initial conditions $\xi(0) = -1$, $\dot{\xi}(0) = 0$. We have

$$\left.\begin{aligned} \xi &= \xi_0 + \xi_1\mu + \xi_2\mu^2 + \ldots., \\ Q^2 &= 1 + Q_2\mu^2 + Q_4\mu^4 + \ldots., \\ P &= 1 + P_1\xi\mu + P_2\xi^2\mu^2 + \ldots., \\ \frac{\partial P}{\partial \xi} &= P_1\mu + 2P_2\xi\mu^2 + \ldots. + nP_n\xi^{n-1}\mu^n + \ldots., \end{aligned}\right\} \tag{32}$$

where the P_n and Q_n are known constants, and the ξ_n are functions of τ to be determined. Substituting (32) in (31) and equating to zero the coefficients of various powers of μ, we find

$$\left.\begin{aligned} &\ddot{\xi}_0 + \xi_0 = 0, \\ &\ddot{\xi}_1 + \xi_1 = \tfrac{1}{2}P_1 - \tfrac{3}{2}P_1\xi_0^2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &\ddot{\xi}_n + \xi_n = R_n(\xi_0, \ldots., \xi_{n-1}), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}\right\} \tag{33}$$

where R_n is a known polynomial in $\xi_0, \ldots., \xi_{n-1}$.

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* *Les Methodes Nouvelles de la Mécanique Céleste,* Vol. I, p. 58.

Equations (33) can be integrated in the order in which they are written. Since the initial conditions are identities in μ, we have

$$\left.\begin{aligned} &\xi_0(0) = -1, \quad \xi_j(0) = 0, \quad j = 1, \dots, \infty, \\ &\dot{\xi}_j(0) = 0, \quad j = 0, \dots, \infty. \end{aligned}\right\} \tag{34}$$

With these initial conditions the solution of the first equation of (33) is

$$\xi_0 = -\cos\tau. \tag{35}$$

Then the second equation of (33) becomes

$$\ddot{\xi}_1 + \xi_1 = -\tfrac{1}{4} P_1 - \tfrac{3}{4} P_1 \cos 2\tau,$$

whose solution satisfying (34) is

$$\xi_1 = -\tfrac{1}{4} P_1 + \tfrac{1}{4} P_1 \cos 2\tau. \tag{36}$$

Let us suppose that $\xi_0, \dots, \xi_{n-1}$ have been computed and that it has been found that ξ_j, $j = 0, \dots, n-1$, is a sum of cosines of multiples of τ, the highest multiple being $j+1$, and then consider the right member of the $(n+1)$-st equation of (33). Its general term, aside from a constant coefficient, has the form

$$[R_n] = \xi_{j_1}^{i_1} \xi_{j_2}^{i_2} \cdots \xi_{j_n}^{i_n}. \tag{37}$$

It follows from (31) and (32) that the exponents and subscripts satisfy the relations

$$\left.\begin{aligned} &i_1 + i_2 + \cdots + i_n = I_n \leqq n + 1, \\ &i_1 j_1 + i_2 j_2 + \cdots + i_n j_n \leqq n - (I_{n-1}). \end{aligned}\right\} \tag{38}$$

Each element of the general term being a sum of cosines, the product is a sum of cosines. The highest multiple is

$$N_n = i_1(j_1 + 1) + i_2(j_2 + 1) + \cdots + i_n(j_n + 1),$$

which, by (38), has as its greatest value $n + 1$. Therefore, the differential equation at the $(n + 1)$-st step is

$$\ddot{\xi}_n + \xi_n = A_0^{(n)} + A_1^{(n)} \cos\tau + A_2^{(n)} \cos 2\tau + \cdots + A_{n+1}^{(n)} \cos(n+1)\tau. \tag{39}$$

The solution of (31) is periodic with the period 2π in τ for all $|\mu|$ sufficiently small. Therefore,

$$\sum_{j=0}^{\infty} \xi_j(\tau + 2\pi)\, \mu^j \equiv \sum_{j=0}^{\infty} \xi_j(\tau)\, \mu^j,$$

where the identity holds in both μ and τ. It follows from the fact that it is an identity in μ that

$$\xi_j(\tau + 2\pi) \equiv \xi_j(\tau), \; j = 0, \dots, \infty. \tag{40}$$

Since this relation is an identity in τ, each ξ_j is separately periodic with the period 2π.

Now consider the integration of (39). Since the solution is periodic, the coefficient $A_1^{(n)}$ is zero. This condition has been secured at each step by the factor $Q^2(\mu)$ in the second of (31). With $A_1^{(n)}=0$, the solution of (39) satisfying (34) is

$$\left.\begin{aligned} &\xi_n = \alpha_0^{(n)} + \alpha_1^{(n)}\cos\tau + \alpha_2^{(n)}\cos 2\tau + \ldots\ldots + \alpha_{n+1}^{(n)}\cos(n+1)\tau, \\ &\alpha_0^{(n)} = A_0^{(n)}, \\ &\alpha_j^{(n)} = \frac{A_j^{(n)}}{1-j^2},\ j=2,\ \ldots.,\ n+1, \\ &\alpha_1^{(n)} = -A_0^{(n)} + \sum_{j=2}^{n+1}\frac{A_j^{(n)}}{j^2-1}. \end{aligned}\right\} \quad (41)$$

These results have the properties assumed in the beginning, and the induction is therefore complete.

§ 6. DIRECT CONSTRUCTION OF THE SOLUTION FROM THE INTEGRAL.

The properties of the solution being known, it can conveniently be found from the first equation of (31). Let this equation be written in the form

$$\dot{\xi}^2 - Q^2(\mu)(1-\xi^2)P(\xi;\mu) \equiv F(\dot{\xi},\xi;\mu) = 0. \qquad (42)$$

This equation holds identically in μ and τ. Since ξ and $\dot{\xi}$ are expansible as converging power-series in μ, F can be expanded as a power-series in μ of the form

$$F = F_0 + F_1\mu + \ldots\ldots + F_n\mu^n + \ldots\ldots = 0.$$

Since this equation is an identity in μ, we have

$$F_n(\dot{\xi}_j,\ \xi_j) = 0,\ j=0,\ \ldots.,\ n;\ n=0,\ \ldots.,\infty. \qquad (43)$$

Now suppose the expressions for ξ_j and $\dot{\xi}_j$ given in the first of (41) are substituted in (43). It follows from (42) that the largest value of j in F_n is $j=n$. The result of the substitution will be a sum of cosines of integral multiples of τ, the highest being $n+2$. Therefore, (43) may be written

$$F_n = B_0^{(n)} + B_1^{(n)}\cos\tau + \ldots\ldots + B_{n+2}^{(n)}\cos(n+2)\tau = 0. \qquad (44)$$

Since this equation is an identity in τ, we have

$$B_j^{(n)} = 0,\ \ j=0,\ \ldots.,\ n+2. \qquad (45)$$

It follows from the form of (42) and the fact that in $P(\xi,\mu)$ every term except the first, which is unity, is multiplied by μ that the $B_j^{(n)}$ have the form

$$\left.\begin{aligned} B_0^{(n)} &= -\,2\alpha_1^{(n)} + C_0^{(n)}, \\ B_1^{(n)} &= -\,2\alpha_0^{(n)} - 3\alpha_3^{(n)} + C_1^{(n)}, \\ B_2^{(n)} &= +\,0 - 4\alpha_3^{(n)} + C_2^{(n)}, \\ B_j^{(n)} &= +\,(j-2)\alpha_{j-1}^{(n)} - (j+2)\alpha_{j+1}^{(n)} + C_j^{(n)},\quad j=2,\ \ldots.,\ n, \\ B_{n+1}^{(n)} &= +\,(n-1)\alpha_n^{(n)} + 0 + C_{n+1}^{(n)}, \\ B_{n+2}^{(n)} &= +\,n\alpha_{n+1}^{(n)} + 0 + C_{n+2}^{(n)}, \end{aligned}\right\} \quad (46)$$

where the $C_j^{(n)}$, $j = 0, \ldots, n + 2$, are known functions of $\alpha_i^{(0)}, \ldots, \alpha_i^{(n-1)}$. Therefore if $\alpha_i^{(0)}, \ldots, \alpha_i^{(n-1)}$ have been computed, equations (45) determine $\alpha_0^{(n)}, \ldots, \alpha_{n+1}^{(n)}$. The solution is most conveniently made by starting with the last equation and using them in order to the first, with the exception of the third, which is redundant. If the constant of integration in (42) had not been determined on the start so as to belong to the solution with the period 2π, the coefficients of the $(n + 1)$-st term of its expansion as a power-series in μ would be determined by the third equation of (46).

§ 7. DIRECT CONSTRUCTION OF THE SOLUTION FROM THE VALUES OF ξ AND ITS DERIVATIVES AT $\tau = 0$ AND $\tau = \pi$.

We may write (31) and their successive derivatives in the form

$$\left.\begin{aligned} \dot{\xi} &= R_1(\xi, \mu), \\ \ddot{\xi} &= R_2(\xi, \mu), \\ &\cdots\cdots\cdots\cdots, \\ \overset{n}{\xi} &= R_n(\xi, \mu), \end{aligned}\right\} \tag{47}$$

where the $R_j(\xi, \mu)$ are power-series in ξ and μ. The derivatives obtained in the right member are of course eliminated at the successive steps by the first equation. Since $\xi = -1$ at $\tau = 0$ and $\xi = +1$ at $\tau = \pi$ identically in μ, we have

$$\left.\begin{aligned} &\xi(0) = \sum_{j=0}^{\infty} \xi_j(0)\mu^j = -1, \\ &\xi(\pi) = \sum_{j=0}^{\infty} \xi_j(\pi)\mu^j = +1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ &\overset{n}{\xi}(0) = \sum_{j=0}^{\infty} \overset{n}{\xi}_j(0)\mu^j = R_n(-1, \mu) = \sum_{j=0}^{\infty} p_n^{(j)}(-1)\mu^j, \quad n = 1, \ldots, \infty, \\ &\overset{n}{\xi}(\pi) = \sum_{j=0}^{\infty} \overset{n}{\xi}_j(\pi)\mu^j = R_n(+1, \mu) = \sum_{j=0}^{\infty} p_n^{(j)}(+1)\mu^j. \end{aligned}\right\} \tag{48}$$

Now consider the determination of the coefficients of ξ_j. Its form is given in (41), from which it follows that all odd derivatives vanish at both $\tau = 0$ and $\tau = \pi$. All $R_{2k+1}(\xi, \mu)$ also vanish at $\tau = 0$ and $\tau = \pi$, because they carry $\sqrt{1 - \xi^2}$ as a factor. Hence the coefficients of equations (48) are identically zero when n is odd. Since equations (48) hold identically in μ, we have for even values of n

$$\begin{aligned}
&\xi_j(0) = 0, \quad j = 1, \ldots, \infty, \\
&\xi_j(\pi) = 0, \\
&\overset{2n}{\xi}_j(0) = p_{2n}^{(j)}(-1), \quad j = 0, \ldots, \infty, \\
&\overset{2n}{\xi}_j(\pi) = p_{2n}^{(j)}(+1);
\end{aligned}$$

or, making use of (31),

$$\left.\begin{aligned}
&\alpha_0^{(j)} + \alpha_1^{(j)} + \ldots + \alpha_{j+1}^{(j)} = 0, \\
&\alpha_0^{(j)} - \alpha_1^{(j)} + \ldots + (-1)^{j+1}\alpha_{j+1}^{(j)} = 0, \\
&0 \quad + \alpha_1^{(j)} + 2^{2n}\alpha_2^{(j)} + \ldots + (j+1)^{2n}\alpha_{j+1}^{(j)} = p_{2n}^{(j)}(-1), \quad n = 1, 2, \ldots, \infty, \\
&0 \quad - \alpha_1^{(j)} + 2^{2n}\alpha_2^{(j)} - \ldots + (-1)^{j+1}(j+1)^{2n}\alpha_{j+1}^{(j)} = p_{2n}^{(j)}(+1).
\end{aligned}\right\} \quad (49)$$

Since the right members of these equations are known, the $\alpha_k^{(j)}$ are uniquely determined by this infinite set of equations, provided the determinant of the coefficients of any $j + 2$ of them is distinct from zero. It is clearly necessary to use at least one of the first two equations, which alone involve $\alpha_0^{(j)}$. If those equations are taken for which $\tau = 0$, the determinant is an alternant and is distinct from zero. Therefore the solutions are uniquely defined by the properties which have been established and by these equations. This is the simplest method of computing them. The results of the computation are easily checked by substituting them in any of the infinitely many superfluous equations.

§ 8. APPLICATION TO THE LEGENDRE ELLIPTIC FUNCTIONS.

The results which have been obtained in §§ 5–7 include the Legendre Elliptic Functions as simple special cases. Consider sn τ, the *sine amplitude* function, which is defined by

$$\left.\begin{aligned}
x' &= \sqrt{(1 - x^2)(1 - \kappa^2 x^2)}, \quad \kappa^2 < 1, \\
x'' &= -(1 + \kappa^2)x + 2\kappa^2 x^3.
\end{aligned}\right\} \quad (50)$$

The parameter κ^2 plays the rôle of μ above, and the solutions are in powers of κ^2. In order that the results may agree with the usual notations, we take the initial condition $x(0) = 0$. In this case we have

$$Q(\mu) = Q(\kappa^2) = 1 + (\tfrac{1}{2})^2\kappa^2 + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2\kappa^4 + \ldots + \left(\frac{1\,.\,3\ldots 2n-1}{2\,.\,4\ldots 2n}\right)^2\kappa^{2n} + \ldots \quad (51)$$

Then

$$\dot{x} = Q(\kappa^2)\sqrt{(1 - x^2)(1 - \kappa^2 x^2)}.$$

The solution of this equation as a power-series in κ^2 is

$$x = x_0 + x_2\kappa^2 + x_4\kappa^4 + \cdots\cdots \tag{52}$$

Making the computation by the methods of §§ 5–7, it is found that

$$\left.\begin{aligned}
x_0 &= \sin\tau,\\
x_2 &= \tfrac{1}{16}\,[\sin\tau + \sin 3\tau],\\
x_4 &= \tfrac{1}{16^2}\,[7\sin\tau + 8\sin 3\tau + \sin 5\tau],\\
x_6 &= \tfrac{1}{16^3}\,[67\sin\tau + 82\sin 3\tau + 16\sin 5\tau + \sin 7\tau],\\
x_8 &= \tfrac{1}{16^4}\,[738\sin\tau + 945\sin 3\tau + 230\sin 5\tau + 24\sin 7\tau + \sin 9\tau],\\
x_{10} &= \tfrac{1}{16^5}\,[8808\sin\tau + 11{,}661\sin 3\tau + 3264\sin 5\tau + 442\sin 7\tau\\
&\qquad\qquad + 32\sin 9\tau + \sin 11\tau],\\
x_{12} &= \tfrac{1}{16^6}\,[110{,}728\sin\tau + 150{,}486\sin 3\tau + 46{,}519\sin 5\tau\\
&\qquad\qquad + 7440\sin 7\tau + 718\sin 9\tau + 40\sin 11\tau + \sin 13\tau],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{53}$$

Each of these expressions carries $\sin\tau$ as a factor. Since $x(\pi/2) = 1$ for all κ^2, it follows that $x_{2n}(\pi/2) = 0$, $n = 1, \ldots, \infty$. Therefore each $x_{2n}(\pi/2)$, $n = 1, \ldots, \infty$, carries $\cos\tau$ as a factor. Since $\dot{x}(\pi/2)$ also vanishes for $\kappa^2 < 1$, it follows that each x_{2n} has $\cos^2\tau$ as a factor. After $\sin\tau\cos^2\tau$ is removed, the result can conveniently be expressed as a sum of even powers of cosines of τ. The explicit results are

$$\left.\begin{aligned}
x_0 &= \sin\tau,\\
x_2 &= \tfrac{1}{4}\sin\tau\cos^2\tau,\\
x_4 &= \tfrac{1}{64}\sin\tau\cos^2\tau\,[5 + 4\cos^2\tau],\\
x_6 &= \tfrac{1}{256}\sin\tau\cos^2\tau\,[10 + 11\cos^2\tau + 4\cos^4\tau],\\
x_8 &= \tfrac{1}{128^2}\sin\tau\cos^2\tau\,[389 + 500\cos^2\tau + 272\cos^4\tau + 64\cos^6\tau],\\
x_{10} &= \tfrac{1}{256^2}\sin\tau\cos^2\tau\,[1054 + 1499\cos^2\tau + 984\cos^4\tau + 368\cos^6\tau\\
&\qquad\qquad + 64\cos^8\tau],\\
x_{12} &= \tfrac{1}{1024^2}\sin\tau\cos^2\tau\,[12{,}242 + 18{,}759\cos^2\tau + 13{,}800\cos^4\tau\\
&\qquad\qquad + 6448\cos^6\tau + 1856\cos^8\tau + 256\cos^{10}\tau],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{54}$$

These expressions are very convenient for use in practical computations.

Let $y = \operatorname{cn}\tau$, the *cosine amplitude* function. Then we have

$$y = \sqrt{1 - x^2} = \sqrt{1 - x_0^2}\,\sqrt{1 - \frac{\left[2x_0 + \sum\limits_{j=1}^{\infty} x_{2j}\kappa^{2j}\right]\sum\limits_{j=1}^{\infty} x_{2j}\kappa^{2j}}{1 - x_0^2}}. \tag{55}$$

It follows from this equation that y is expansible as a power-series in κ^2 and that the coefficient of κ^{2n} is a sum of cosines of odd multiples of τ, the highest being $2n+1$. The explicit expression for y can be derived from this equation, but it is just as conveniently found from the differential equation which y satisfies, viz.,

$$\dot{y} = Q(\kappa^2)\sqrt{(1-y^2)(1-\kappa^2+\kappa^2 y^2)}, \tag{56}$$

where $y(0)=1$, $y(\pi/2)=0$. Forming the successive derivatives, equations corresponding to (49) are obtained. Since $y(\pi/2)\equiv 0$, it follows that each y_{2n}, $n=0, \ldots, \infty$, carries $\cos\tau$ as a factor, and since $y(0)\equiv 1$, $\dot{y}(0)\equiv 0$, it follows that each y_{2n}, $n=1, \ldots, \infty$, carries $\sin^2\tau$ as a factor. These results also follow directly from the properties of x and the form of (55). Making the computation, we find

$$\left.\begin{aligned}
y &= y_0 + y_2\kappa^2 + y_4\kappa^4 + \cdots,\\
y_0 &= \cos\tau,\\
y_2 &= -\tfrac{1}{4}\cos\tau\sin^2\tau,\\
y_4 &= -\tfrac{1}{64}\cos\tau\sin^2\tau\,[7+4\cos^2\tau],\\
y_6 &= -\tfrac{1}{256}\cos\tau\sin^2\tau\,[17+13\cos^2\tau+4\cos^4\tau],\\
y_8 &= -\tfrac{1}{128^2}\cos\tau\sin^2\tau\,[759+660\cos^2\tau+304\cos^4\tau+64\cos^6\tau],\\
y_{10} &= -\tfrac{1}{256^2}\cos\tau\sin^2\tau\,[2289+2149\cos^2\tau+1192\cos^4\tau+400\cos^6\tau\\
&\qquad +64\cos^8\tau],\\
y_{12} &= -\tfrac{1}{1024^2}\cos\tau\sin^2\tau\,[29{,}023+28{,}727\cos^2\tau+17{,}800\cos^4\tau\\
&\qquad +7472\cos^6\tau+1984\cos^8\tau+256\cos^{10}\tau],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{57}$$

Now let $z=\sqrt{1-\kappa^2 x^2}$, the *delta amplitude* function. The variable z satisfies the equation

$$\dot{z} = Q(\kappa^2)\sqrt{(1-z^2)(-1+\kappa^2+z^2)}. \tag{58}$$

Making the computations as before, we find for the explicit value of z

$$\left.\begin{aligned}
z &= 1 + z_2\kappa^2 + z_4\kappa^4 + \cdots,\\
z_2 &= -\tfrac{1}{2}\sin^2\tau,\\
z_4 &= -\tfrac{1}{8}\sin^2\tau\,[1+\cos^2\tau],\\
z_6 &= -\tfrac{1}{64}\sin^2\tau\,[4+5\cos^2\tau+2\cos^4\tau],\\
z_8 &= -\tfrac{1}{32^2}\sin^2\tau\,[40+56\cos^2\tau+32\cos^4\tau+8\cos^6\tau],\\
z_{10} &= -\tfrac{1}{128^2}\sin^2\tau\,[448+677\cos^2\tau+454\cos^4\tau+176\cos^6\tau\\
&\qquad +32\cos^8\tau],\\
z_{12} &= -\tfrac{1}{256^2}\sin^2\tau\,[1344+2152\cos^2\tau+1586\cos^4\tau+756\cos^6\tau\\
&\qquad +224\cos^8\tau+32\cos^{10}\tau],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{59}$$

The functions $\operatorname{sn}\tau$, $\operatorname{cn}\tau$ and $\operatorname{dn}\tau$ arise very naturally in many problems of mechanics and geometry.* The series (54), (57) and (59) are convenient for use in solving these problems, and the coefficients which are given here can be relied upon as being correct. But the point of chief interest is that if the differential equations defining the functions are of the more general type considered here, the same methods of obtaining their explicit values can be used.

§ 9. THE REALM OF CONVERGENCE OF THE SERIES (52), (57) AND (59).

For simplicity let us first consider the series for the sine amplitude function, which is defined by

$$\left.\begin{aligned} \dot{x} &= Q(\kappa^2)\sqrt{(1-x^2)(1-\kappa^2x^2)}, \quad x = 0 \text{ at } \tau = 0, \\ \ddot{x} &= -Q^2(\kappa^2)(1+\kappa^2)x + 2\,Q^2(\kappa^2)\kappa^2x^3. \end{aligned}\right\} \quad (60)$$

It follows from the second of these equations and the fact that $Q(\kappa^2)$ converges for all $|\kappa^2| < 1$ that the solution for x considered as a function of κ^2 is regular for all $|\kappa^2| < 1$ and all finite values of τ, provided x does not become infinite for some value of τ. In the solutions (52) and (54) τ enters as a parameter and takes only real values. Therefore, if x does not become infinite for any real τ on the interval 0 to 2π, the solution (52), being periodic with the period 2π, converges for all $|\kappa^2| < 1$ and all finite real values of τ.

In order to establish the result that x does not become infinite for a real value of τ, we shall show that the integral

$$I = \int_0^\infty \frac{dx}{Q(\kappa^2)\sqrt{(1-x^2)(1-\kappa^2x^2)}}$$

is not real. It is necessary to specify the path of integration from 0 to ∞. We shall integrate from zero to $1-\epsilon$ along the real axis; then along the circle with radius ϵ and center at 1 to the line joining 1 with the point $1/\kappa$, which will in general be complex; then along the line from 1 to $1/\kappa$ to a distance ϵ from $1/\kappa$; then along the small circle ϵ around $1/\kappa$ to the line from $1/\kappa$ to ∞ whose inclination to the axis of reals is the argument of $1/\kappa$; then along this line to ∞. This may be indicated by

$$I = I_1 + I_2 + I_3 + I_4 + I_5 = \int_0^1 + \int_{\epsilon(1)} + \int_1^{1/\kappa} + \int_{\epsilon(1/\kappa)} + \int_{1/\kappa}^\infty, \qquad (61)$$

where $I_1, \ldots, I_5$ represent the integrals (61) in order. It is easy to show that $\lim\limits_{\epsilon=0} I_2 = \lim\limits_{\epsilon=0} I_4 = 0$. The transformation $x = 1/\kappa\xi$ throws I_5 into the form I_1.

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* See Appell and Lacour, *Fonctions Elliptiques*, Chap. V.

Therefore we have

$$I = 2\int_0^1 \frac{dx}{Q(\kappa^2)\sqrt{(1-x^2)(1-\kappa^2x^2)}} + \int_1^{1/\kappa} \frac{dx}{Q(\kappa^2)\sqrt{(1-x^2)(1-\kappa^2x^2)}}. \qquad (62)$$

By the definition of $Q(\kappa^2)$ we have

$$\int_0^1 \frac{dx}{Q(\kappa^2)\sqrt{(1-x^2)(1-\kappa^2x^2)}} = \frac{\pi}{2}\,\frac{Q(\kappa^2)}{Q(\kappa^2)} = \frac{\pi}{2},$$

which is real whatever the value of κ^2. It is one-fourth of one period of the elliptic function. The other integral is one-half of the other period, and therefore it can not be real, since the ratio of the two periods can not be real.

Now consider any other path of integration to infinity. If none of the branch-points ± 1, $\pm 1/\kappa$ is included between the two paths of integration, the value of the integral is the same. Suppose only the branch-point $+1$ is included between the two paths. Then the new path can be deformed, without crossing a branch-point, into the old one plus an integral from zero to $+1$ and back, after making the circuit of this point. Hence, in this case the value of the integral is $4I_1 + I_3$, which is not real. Suppose only the branch-point $1/\kappa$ is included between the two paths. The new path can be deformed, without crossing a branch-point, into the old one plus an integral from ∞ to $1/\kappa$ and back, after a circuit around $1/\kappa$, along a line whose inclination to the axis of reals is the argument of $1/\kappa$. Hence, in this case the value of the integral is $2I_1 + I_3 - 2I_1$, which is not real. It is easy to see that the integral from zero to infinity along any path is $2I_1 + I_3 + 2m_1I_1 + 2m_3I_3$, where m_1 and m_3 are positive or negative integers, or zero. This sum can not be real and it follows that x can not become infinite for any real finite τ. *Therefore, the series* (52) *converges for all real finite values of* τ, *provided* $|\kappa^2| < 1$.

An analogous discussion proves the same result for the expansions of the cosine amplitude and the delta amplitude functions.

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